

Convex Bodies With Minimal Volume Product in \mathbb{R}^2 — A New Proof

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Abstract. In this paper, a new proof of the following result is given: The product of the volumes of an origin symmetric convex bodies K in \mathbb{R}^2 and of its polar body is minimal if and only if K is a parallelogram.

1. Introduction

A well-known problem in the theory of convex sets is to find a lower bound for the product of volumes $\mathcal{P}(K) = V(K)V(K^*)$, which is called *the volume-product of K* , where K is an n -dimensional origin symmetric convex body and K^* is the polar body of K (see definition in Section 2). Is it true that we always have

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n), \quad (1.1)$$

where $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leq 1, 1 \leq i \leq n\}$?

For some particular classes of convex symmetric bodies in \mathbb{R}^n , a sharper estimate for the lower bound of $\mathcal{P}(K)$ has been obtained. If K is the unit ball of a normed n -dimensional space with a 1-unconditional basis, J. Saint-Raymond [12] proved that $\mathcal{P}(K) \geq 4^n/n!$; the equality case, obtained for $1 - \infty$ spaces, is discussed in [6] and [11]. When K is a zonoid it was proved in [2] and [10] that the same inequality holds, with equality if and only if K is an n -cube.

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In [1], J. Bourgain and V. D. Milman proved that there exist some $c > 0$ such that for every n and every convex body K of \mathbb{R}^n ,

$$\mathcal{P}(K) \geq c^n \mathcal{P}(B_2^n).$$

The best known constant $c = \frac{\pi}{4}$ is due to Kuperberg [3].

In [4], K. Mahler proved (1.2) when $n = 2$. There are several other proofs of the two-dimensional result, see for example the proof of M. Meyer, [7], but the question is still open even in the three-dimensional case.

In this paper, we present a new proof about the problem when $n = 2$, which is different from the proof in [4] and [7]. Firstly, we prove that any origin symmetric polygon satisfies the conjecture. Then, using the continuity of $\mathcal{P}(K)$ with respect to the Hausdorff metric, we can easily prove that the conjecture is also correct for any origin symmetric convex bodies in \mathbb{R}^2 . For the three-dimensional case, the conjecture maybe can be solved by use of the same idea.

Finally, let us mention the problem of giving an upper bound to $\mathcal{P}(K)$; it was proved by L. A. Santaló [13]: $\mathcal{P}(K) \leq \mathcal{P}(B_2^n)$, where B_2^n is the n -dimensional Euclidean unit ball. In [5], [8] and [9], it was shown that the equality holds only if K is an ellipsoid.

2. Notations and background materials

As usual, S^{n-1} denotes the unit sphere, B^n the unit ball centered at the origin, o the origin and $\|\cdot\|$ the norm in Euclidean n -space \mathbb{R}^n . If $x, y \in \mathbb{R}^n$, then $\langle x, y \rangle$ is the inner product of x and y .

If K is a set, ∂K is its boundary, $\text{int } K$ is its interior, and $\text{conv } K$ denotes its convex hull. Let $\mathbb{R}^n \setminus K$ denote the complement of K , i.e., $\mathbb{R}^n \setminus K = \{x \in \mathbb{R}^n : x \notin K\}$. If K is a n -dimensional convex subset of \mathbb{R}^n , then $V(K)$ is its volume $V_n(K)$.

Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with non-empty interiors) in \mathbb{R}^n . Let \mathcal{K}_o^n denote the subset of \mathcal{K}^n that contains the origin in its interior. Let $h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, denote the support function of $K \in \mathcal{K}_o^n$; i.e.,

$$h(K, u) = \max\{u \cdot x : x \in K\}, u \in S^{n-1}, \quad (2.1)$$

and let $\rho(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, denote the radial function of $K \in \mathcal{K}_o^n$; i.e.,

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, u \in S^{n-1}. \quad (2.2)$$

A linear transformation (or affine transformation) of \mathbb{R}^n is a map ϕ from \mathbb{R}^n to itself such that $\phi x = Ax$ (or $\phi x = Ax + t$, respectively), where A is an $n \times n$ matrix and $t \in \mathbb{R}^n$. By definition, for any parallelograms centered at the origin $ABCD$ and $A'B'C'D'$, there always is a linear transformation \mathcal{A} taking $ABCD$ to $A'B'C'D'$.

Geometrically, an affine transformation in Euclidean space is one that preserves:

- (1). The collinearity relation between points; i.e., three points which lie on a line continue to be collinear after the transformation.
- (2) Ratios of distances along a line; i.e., for distinct collinear points P_1, P_2, P_3 , the ratio $|P_2 - P_1|/|P_3 - P_2|$ is preserved.

If $K \in K_o^n$, we define the polar body of K , K^* , by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}.$$

It is easy to verify that (see p.44 in [14])

$$h(K^*, u) = \frac{1}{\rho(K, u)} \quad \text{and} \quad \rho(K^*, u) = \frac{1}{h(K, u)} \quad (2.3)$$

If P is a polygon, i.e., $P = \text{conv}\{p_1, \dots, p_m\}$, where p_i ($i = 1, \dots, m$) are vertices of polygon P . By the definition of polar body, we have

$$\begin{aligned} P^* &= \{x \in \mathbb{R}^2 : x \cdot p_1 \leq 1, \dots, x \cdot p_m \leq 1\} \\ &= \bigcap_{i=1}^m \{x \in \mathbb{R}^2 : x \cdot p_i \leq 1\}, \end{aligned} \quad (2.4)$$

which implies that P^* is the intersection of m closed half-planes with exterior normal vector p_i and the distance of straight line $\{x \in \mathbb{R}^2 : x \cdot p_i = 1\}$ from the origin is $1/\|p_i\|$. Thus, if P is an inscribed polygon in a unit circle, then P^* is polygon circumscribed around the unit circle. In the proof of Lemma 3.3, we shall make use of these properties.

For $K, L \in \mathcal{K}^n$ the Hausdorff distance is defined by

$$d(K, L) = \min\{\lambda \geq 0 : K \subset L + \lambda B^n, L \subset K + \lambda B^n\}, \quad (2.5)$$

which can be conveniently defined by (see p.53 in [14])

$$d(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|, \quad (2.6)$$

therefore, a sequence of convex bodies K_i converges to K if and only if the sequence of support function $h(K_i, \cdot)$ converges uniformly to $h(K, \cdot)$.

In \mathcal{K}_o^n , the convergence of convex bodies is equivalent to the uniform convergence of their radial functions. Because the conclusion will be used in the proof of Lemma 3.5, we prove this conclusion (this proof is due to Professor Zhang Gaoyong and we listened his lecture in Chongqing).

Let $K \in \mathcal{K}_o^n$. Define

$$r_1 = \max_{u \in S^{n-1}} \rho(K, u), \quad (2.7)$$

$$r_0 = \min_{u \in S^{n-1}} \rho(K, u). \quad (2.8)$$

It is easily seen that

$$r_1 = \max_{u \in S^{n-1}} h(K, u), \quad (2.9)$$

$$r_0 = \min_{u \in S^{n-1}} h(K, u). \quad (2.10)$$

Lemma 2.1. If $K \in \mathcal{K}_o^n$, then

$$\rho(K + tB^n, u) \leq \rho(K, u) + \frac{r_1}{r_0}t, \quad (2.11)$$

$$|u \cdot v(x)| \geq \frac{r_0}{r_1}, \quad (2.12)$$

where $x = u\rho(K, u) \in \partial K$.

Proof. For $x \in \partial K$, let x' be the point on $\partial(K + tB^n)$ and has the same direction as x . Let $u = x/\|x\| = x'/\|x'\|$. Then

$$\rho(K + tB^n, u) - \rho(K, u) = \|x' - x\|.$$

Since K and $K + tB^n$ are parallel, the projection length of $x' - x$ onto the normal $v(x)$ is less than t ,

$$\|x' - x\| \leq \frac{t}{|u \cdot v(x)|}.$$

There is

$$\begin{aligned} |u \cdot v(x)| &= \frac{|x \cdot v(x)|}{\|x\|} \\ &= \frac{h(K, v(x))}{\|x\|} \\ &\geq \frac{r_0}{r_1}. \end{aligned} \quad (2.13)$$

The desired inequalities follow. \square

Theorem 2.2. If a sequence of convex bodies $K_i \in \mathcal{K}_0^n$ converges to $K \in \mathcal{K}_0^n$ in the Hausdorff metric, then the sequence of radial functions $\rho(K_i, \cdot)$ converges to $\rho(K, \cdot)$ uniformly.

Proof. Assume that $d(K_i, K) < \varepsilon$. Then $K_i \subset K + \varepsilon B^n$, and $K \subset K_i + \varepsilon B^n$. By Lemma 2.1, (2.9) and (2.10)

$$\rho(K_i, \cdot) \leq \rho(K, \cdot) + \frac{r_1}{r_0} \varepsilon,$$

$$\rho(K, \cdot) \leq \rho(K_i, \cdot) + \frac{r_1 + \varepsilon}{r_0 - \varepsilon} \varepsilon.$$

When $\varepsilon < r_0/2$, we have

$$|\rho(K_i, \cdot) - \rho(K, \cdot)| \leq \frac{4r_1}{r_0} \varepsilon,$$

therefore the sequence of radial functions $\rho(K_i, \cdot)$ converges to $\rho(K, \cdot)$ uniformly. \square

3. Main result and its proof

First, looking the following important theorem:

Theorem 3.1. For any origin symmetric convex body $K \subset \mathbb{R}^n$, $\mathcal{P}(K)$ is linear invariant, that is, for every linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we have $\mathcal{P}(AK) = \mathcal{P}(K)$.

Proof. For any $u \in S^{n-1}$, we have

$$\rho((AK)^*, u) = \frac{1}{h(AK, u)} = \frac{1}{h(K, A^t u)} = \rho(K^*, A^t u) = \rho(A^{-t} K^*, u).$$

Hence, $(AK)^* = A^{-t} K^*$, therefore

$$\begin{aligned} \mathcal{P}(AK) &= V(AK)V((AK)^*) = V(AK)V(A^{-t} K^*) \\ &= |A||A^{-t}|V(K)V(K^*) = V(K)V(K^*) = \mathcal{P}(K). \end{aligned}$$

\square

Because any parallelogram can be linear transformed into a unit square, therefore their volume product is same (this value is equal to 8). By the theorem above, we consider linear transformation of origin symmetric polygon. We obtain the following theorem, which is critical in our proof.

Theorem 3.2. In \mathbb{R}^2 , for any origin symmetric polygon P , there exists a linear transformation $\mathcal{A} : P \rightarrow P'$, where P' satisfies that $P' \subset B^2$ and there exist three continuous vertices contained in ∂B^2 .

Proof. Since P is origin symmetric polygon, its number of sides is an even and corresponding two sides are parallel. Let $A_1, \dots, A_n, B_1, \dots, B_n$ denote all vertices of P . In order to prove this theorem, we need three steps.

The first step, transforming parallelogram $A_1A_2B_1B_2$ into rectangular $A'_1A'_2B'_1B'_2$ inscribed in B^2 . Now P is transformed into P_1 (see (2) or (2)' in Figure 3.1.1 and 3.1.2).

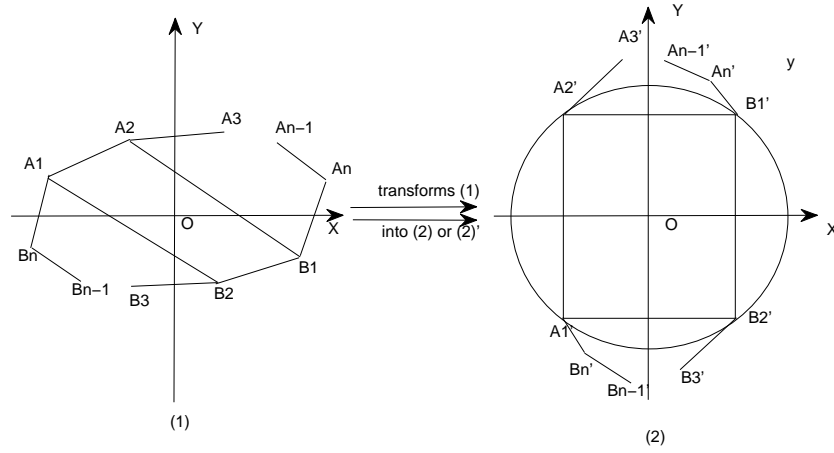


Figure 3.1.1

The second step, transforming P_1 into P_2 (see (3) in Figure 3.1.2). For polygon P_1 , if there exist some vertices

$$\{A'_i : i \in I \subset \{3, \dots, n\}\} \subset \mathbb{R}^2 \setminus B^2,$$

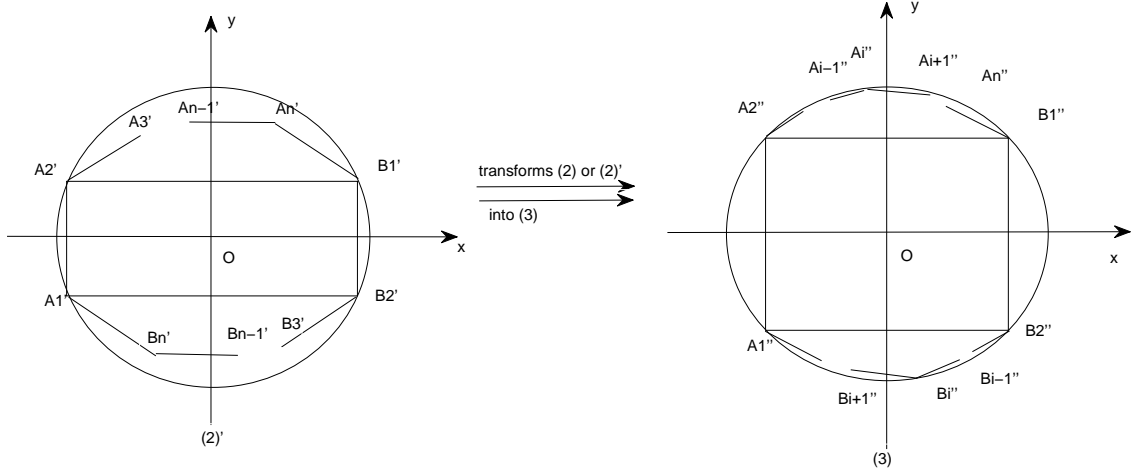


Figure 3.1.2

there exists a linear transformation $\mathcal{A}_1 : P_1 \rightarrow P_2$, which shortens segment $A_1'A_2'$ and $B_1'B_2'$ into $A_1''A_2''$ and $B_1''B_2''$, simultaneously makes some vertices $\{A_i'' : i \in I_2 \subset \{3, \dots, n\}\}$ on boundary of B^2 and $P_2 \subset B^2$. If

$$\{A_3', \dots, A_n', B_3', \dots, B_n'\} \subset \text{int } B^2,$$

then there exists a linear transformation $\mathcal{A}'_1 : P_1 \rightarrow P_2$, which lengthens segments $A_1'A_2'$ and $B_1'B_2'$ into $A_1''A_2''$ and $B_1''B_2''$ respectively, simultaneously makes some vertices $\{A_i'' : i \in I_1 \subset \{3, \dots, n\}\}$ on boundary of B^2 and $P_2 \subset B^2$. (see (3) in Figure 3.1.2).

The third step, transforming P_2 into P_3 . If A_1'', A_2'', A_i'' are three continuous vertices contained in ∂B^2 , then this theorem has been proved; otherwise rotation transforming P_2 into P'_3 , which satisfies that $A_2''A_i''$ parallels x-axis (see (4) in Figure 3.2). Then we transform P'_3 into P_3 , lengthening segments $A_2''B_i''$ and $A_i''B_2''$ into $A_2^{(3)}B_i^{(3)}$ and $A_i^{(3)}B_2^{(3)}$ respectively, simultaneously making some vertices $\{A_j^{(3)} : j \in I_3 \subset \{3, \dots, i-1\}\}$ on boundary of B^2 and $P_3 \subset B^2$ (Since it is easy to prove that vertices $\{A_{i+1}^{(3)}, \dots, A_n^{(3)}, B_1^{(3)}\}$ are in the internal of B^2) (see (5) in Figure 3.2).

Repeating the third step finite times, we can get a polygon P' , in which there exist three continuous vertices contained in ∂B^2 , which completes the

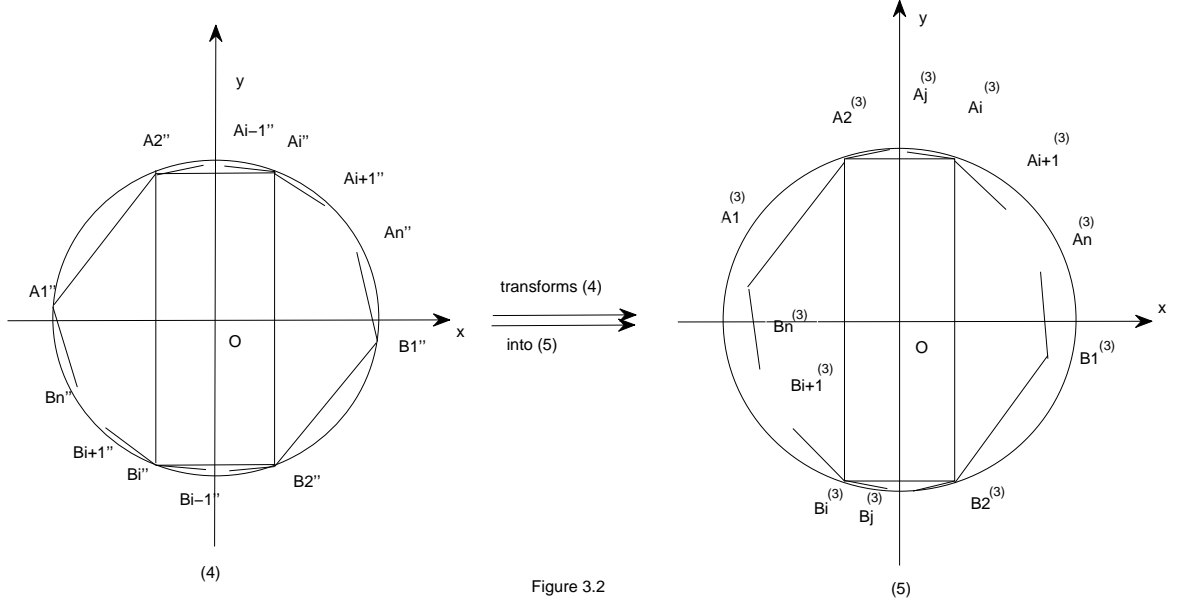


Figure 3.2

proof. \square

By above theorem, we consider the volume-product of polygon with three continuous vertices in ∂B^2 .

Lemma 3.3. Suppose that $P' \subset B^2$ is an origin symmetric polygon and A, C, B are three continuous vertices of P' contained in ∂B^2 , then $\mathcal{P}(P'') \leq \mathcal{P}(P')$, where P'' is a new polygon from P' by deleting vertices C and C' .

Proof. Suppose side AB parallels X-axis (see Figure 3.3.), straight lines l, l_1 and l_2 are tangent lines to the unit circle B^2 passing through points C, A and B respectively. Let $A = (-x_0, y_0)$, then $B = (x_0, y_0)$. Let θ denote $\angle xOC$. It is clear that $\pi/2 \leq \theta \leq \pi - \arctan(y_0/x_0)$ when point C is in third quadrant. We have the following equations of straight lines:

$$\begin{aligned}
 l_1 : \quad y - y_0 &= \frac{x_0}{y_0}(x + x_0), \\
 l_2 : \quad y - y_0 &= -\frac{x_0}{y_0}(x - x_0), \\
 l : \quad y - \sin \theta &= -\frac{\cos \theta}{\sin \theta}(x - \cos \theta).
 \end{aligned}$$

Therefore we can obtain the area of $\triangle MHL$:

$$S_{\triangle MHL} = \frac{x_0}{y_0} \cdot \frac{\sin \theta - y_0}{\sin \theta + y_0}.$$

Let $V = V(P'')$ and $V^0 = V(P''^*)$, where P'' denotes the new polygon from P' by deleting vertices C and C' , then $\mathcal{P}(P')$ is a function $f(\theta)$, where

$$f(\theta) = (V + 2x_0(\sin \theta - y_0)) \left(V^0 - \frac{2x_0}{y_0} \cdot \frac{\sin \theta - y_0}{\sin \theta + y_0} \right) \quad (3.3)$$

and

$$\frac{\pi}{2} \leq \theta \leq \pi - \arctan\left(\frac{y_0}{x_0}\right).$$

We have

$$f'(\theta) = 2x_0 \cos \theta \cdot \frac{(V^0 y_0 - 2x_0)(\sin \theta + y_0)^2 + 2y_0(4x_0 y_0 - V)}{y_0(\sin \theta + y_0)^2}. \quad (3.4)$$

In (3.6), since $\cos \theta \leq 0$ and $y_0(\sin \theta + y_0)^2 \geq 0$, in order to prove $f'(\theta) \leq 0$, let $t = \sin \theta$, we just need to prove $g(t) \geq 0$, where

$$g(t) = (V^0 y_0 - 2x_0)(t + y_0)^2 + 2y_0(4x_0 y_0 - V), \quad t \in [y_0, 1]. \quad (3.5)$$

In order to prove $g(t) \geq 0$, we just need to prove that $V^0 y_0 - 2x_0 > 0$ and $g(y_0) \geq 0$.

Because of $V(P''^*) \geq V(\text{conv}\{A, M, B, A', M', B'\})$,

$$V^0 \geq 4x_0 y_0 + 2x_0 \left(\frac{1}{y_0} - y_0 \right),$$

therefore,

$$\begin{aligned} V^0 y_0 - 2x_0 &\geq \left(4x_0 y_0 + 2x_0 \left(\frac{1}{y_0} - y_0 \right) \right) y_0 - 2x_0 \\ &= 2x_0 y_0^2 \\ &> 0, \end{aligned} \quad (3.6)$$

and therefore function $g(t)$ is a parabola opening upward. Thence, when $t \in [y_0, 1]$, quadratic function $g(t)$ is increasing, thus we just need to proof

$$g(y_0) = 2y_0(2V^0 y_0^2 - V) \geq 0. \quad (3.7)$$

Let \mathcal{D} denote the area of circular segment enclosed by arc $\widehat{BA'}$ and chord $\overline{BA'}$, then

$$V^0 \geq 4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right) + 2\mathcal{D} \quad (3.8)$$

and

$$V \leq 4x_0y_0 + 2\mathcal{D}. \quad (3.9)$$

In order to prove (3.9), we just need to prove

$$2\left(4x_0y_0 + 2x_0\left(\frac{1}{y_0} - y_0\right) + 2\mathcal{D}\right)y_0^2 \geq 4x_0y_0 + 2\mathcal{D}, \quad (3.10)$$

which equivalent to

$$2x_0y_0^3 \geq \mathcal{D}(1 - 2y_0^2). \quad (3.11)$$

And because

$$\mathcal{D} \leq (1 - x_0) \cdot 2y_0, \quad (3.12)$$

hence, we just need to prove

$$x_0y_0^3 \geq y_0(1 - x_0)(1 - 2y_0^2), \quad (3.13)$$

which equivalent to

$$x_0^3 - 2x_0^2 + 1 \geq 0, \quad (3.14)$$

which is clearly correct.

Summary, we get $f'(\theta) \leq 0$ when $\theta \in [\pi/2, \pi - \arctan(y_0/x_0)]$, hence when $\theta = \pi - \arctan(y_0/x_0)$, which implies that point C coincides with point A , function $f(\theta)$ obtain minimal function value, therefore $\mathcal{P}(P'') \leq \mathcal{P}(P')$. \square

Making use of Lemma 3.3, we can obtain the following conclusion.

Theorem 3.4. If $P \subset \mathbb{R}^2$ is an origin symmetric polygon, then $\mathcal{P}(P) \geq \mathcal{P}(S)$, where S is square.

Proof. By Theorem 3.2, Lemma 3.3 and linear invariance of $\mathcal{P}(P)$, if the number of sides of polygon P is $2n$, there exists a polygon P_1 with $2(n-1)$ sides satisfying $\mathcal{P}(P_1) \leq \mathcal{P}(P)$. Repeating this process $n-2$ times, we can

obtain a square S satisfying $\mathcal{P}(P) \geq \mathcal{P}(S)$. \square

In order to obtain the main result in the paper, we first prove the following lemma.

Lemma 3.5. The volume product $\mathcal{P}(K)$ is continuous under the Hausdorff metric.

Proof. Let

$$\lim_{i \rightarrow \infty} K_i = K.$$

By Theorem 2.2, the sequence of radial function $\rho(K_i, \cdot)$ converges to $\rho(K, \cdot)$ uniformly, therefore the reciprocal of radial function $1/\rho(K_i, \cdot)$ converges to $1/\rho(K, \cdot)$ uniformly. Since

$$\begin{aligned} d(K_i^*, K^*) &= \max_{u \in S^{n-1}} |h(K_i^*, u) - h(K^*, u)| \\ &= \max_{u \in S^{n-1}} \left| \frac{1}{\rho(K_i, u)} - \frac{1}{\rho(K, u)} \right|, \end{aligned} \quad (3.15)$$

we have

$$\lim_{i \rightarrow \infty} K_i^* = K^*. \quad (3.16)$$

By continuity of the volume function $V(\cdot)$ under the Hausdorff metric, we have

$$\begin{aligned} \mathcal{P}(K) &= V(K)V(K^*) \\ &= \lim_{i \rightarrow \infty} V(K_i) \lim_{i \rightarrow \infty} V(K_i^*) \\ &= \lim_{i \rightarrow \infty} V(K_i)V(K_i^*) \\ &= \lim_{i \rightarrow \infty} \mathcal{P}(K_i). \end{aligned} \quad (3.17)$$

\square

Theorem 3.6. If $K \subset \mathbb{R}^2$ is an origin symmetric convex body and $S \subset \mathbb{R}^2$ is a square, then $\mathcal{P}(K) \geq \mathcal{P}(S)$.

Proof. For any origin symmetric convex body $K \subset \mathbb{R}^2$, there exists a sequence of origin symmetric polytopes $\{P_i\}$ converging to K under the Hausdorff metric. By Theorem 3.4 and Lemma 3.5, we have

$$\mathcal{P}(K) = \lim_{n \rightarrow \infty} \mathcal{P}(P_i) \geq \mathcal{P}(S). \quad (3.18)$$

\square

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